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Applied Mathematics Letters 21 (2008) 1118–1123
Applied Mathematics Letters
www.elsevier.com/locate/aml

Eventually positive and bounded solutions of even-order nonlinear neutral differential equations

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Received 27 December 2005; received in revised form 1 December 2007; accepted 11 December 2007

Abstract

Consider the even-order nonlinear neutral differential equation

$$\left[a(t)x(t)-\sum_{i=1}^{m} b_i(t)x(t-\tau_i)\right]^{(n)}-\sum_{j=1}^{l} p_j(t)f_j\left(t,x(t-\sigma_j)\right)=0,\quad t\geq 0,$$

and the associated differential inequality

$$\left[a(t)x(t)-\sum_{i=1}^{m} b_i(t)x(t-\tau_i)\right]^{(n)}-\sum_{j=1}^{l} p_j(t)f_j\left(t,x(t-\sigma_j)\right)\geq 0,\quad t\geq 0.$$

Using Lebesgue's dominated convergence theorem, a necessary and sufficient condition for the existence of eventually positive and bounded solutions is obtained.

Keywords: Even order; Neutral differential equation; Nonlinear; Eventually positive and bounded solution; Comparison theorem

1. Introduction and preliminary

In this work, we consider the even-order nonlinear neutral differential equation

$$\left[a(t)x(t)-\sum_{i=1}^{m} b_i(t)x(t-\tau_i)\right]^{(n)}-\sum_{j=1}^{l} p_j(t)f_j\left(t,x(t-\sigma_j)\right)=0,\quad t\geq 0,\tag{1}$$

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0893-9659/$ - see front matter
doi:10.1016/j.aml.2007.12.011

and the associated differential inequality

$$\left[a(t)x(t) - \sum_{i=1}^{m} b_i(t)x(t-\tau_i)\right]^{(n)} - \sum_{j=1}^{l} p_j(t) f_j\left(t, x(t-\sigma_j)\right) \geq 0, \quad t \geq 0, \tag{2}$$

where $n = 2k, k > 0$ is an integer, $\tau_i, \sigma_j \geq 0, a, b_i, p_j \in C\left([0,\infty), R^+\right)$ $(i = 1, 2, \ldots, m, j = 1, 2, \ldots, l,)$, and $a(t) > 0$, p_j are not identically zero for sufficiently large t, $f_j(t,u)$ are continuously nondecreasing real functions with respect to u defined on R such that $f_j(t,u) > 0$, for $u > 0$, $j = 1, 2, \ldots, l$.

Recently there has been a lot of activity concerning the existence of eventually positive solutions for nonlinear neutral differential equations. See [1–7]. In [1], Zhang has studied the odd-order neutral differential equation

$$[a(t)x(t) - b(t)x(t-\tau)]^{(n)} + p(t) f(x(t-\sigma)) = 0, \quad t \geq 0, \tag{3}$$

and the associated differential inequality

$$[a(t)x(t) - b(t)x(t-\tau)]^{(n)} + p(t) f(x(t-\sigma)) \leq 0, \quad t \geq 0. \tag{4}$$

He has obtained that the existences of eventually positive solutions of (3) and (4) are equivalent. In [2], OuYang has studied the odd-order neutral differential equation

$$\left[a(t)x(t) - \sum_{i=1}^{m} b_i(t)x(t-\tau_i)\right]^{(n)} + \sum_{j=1}^{l} p_j(t) f_j\left(x(t-\sigma_j)\right) = 0, \quad t \geq 0, \tag{5}$$

and the associated differential inequality

$$\left[a(t)x(t) - \sum_{i=1}^{m} b_i(t)x(t-\tau_i)\right]^{(n)} + \sum_{j=1}^{l} p_j(t) f_j\left(x(t-\sigma_j)\right) \leq 0, \quad t \geq 0. \tag{6}$$

He has obtained that the existences of eventually positive solutions of (5) and (6) are equivalent. In [3], Fan and Li have studied the even-order neutral differential equation

$$[a(t)x(t) - b(t)x(t-r)]^{(n)} + q(t) f(x(t-\sigma)) = 0, \quad t \geq 0, \tag{7}$$

and the associated differential inequality

$$[a(t)x(t) - b(t)x(t-r)]^{(n)} + q(t) f(x(t-\sigma)) \leq 0, \quad t \geq 0. \tag{8}$$

where $n = 2k, k > 0$ is an integer, $a, b, q \in \left([0,\infty), R^+\right), r > 0, \sigma \geq 0$, and $a(t) > 0, q$ is not identically zero for sufficiently large t, $f(x)$ is a continuously nondecreasing real function with respect to x defined on R, such that $f(x) > 0$, for $x > 0$. For convenience, we cite the main result of [3] as follows.

Theorem A *([3, Theorem 1]). Suppose there exist $t^* \geq 0, M > 0, r > 0$ such that*

$$\prod_{i=0}^{k} \frac{b(t^*+ir)}{a(t^*+ir)} \leq M \quad (k = 0, 1, 2, \ldots),$$

again set $\mu(t)$ as arbitrarily continuous and eventually positive function, and

$$\lim_{k\to\infty} \left[\frac{Q(t^*+kr)}{a(t^*+kr)} + \frac{b(t^*+kr)Q(t^*+(k-1)r)}{a(t^*+kr)a(t^*+(k-1)r)} + \cdots + \frac{b(t^*+kr)\cdots b(t^*+r)Q(t^*)}{a(t^*+kr)\cdots a(t^*)}\right] = \infty,$$

where

$$Q(t) = Q(t,\mu(t)) = \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} q(s) f(\mu(s-\sigma))\, \mathrm{d}s < \infty,$$

for sufficiently large t. In addition, one of the following conditions holds.

(N_1) $b(t) + \sigma q(t) > 0$, *for sufficiently large* t,

(N_2) $\sigma > 0$, *and* $q(s) \neq 0,\ s \in [t-\sigma, t]$, *for sufficiently large* t.

Then (7) has an eventually positive solution if and only if (8) has an eventually positive solution.

Obviously, it is difficult to verify the conditions of Theorem A. So its applications are restricted. Our aim in this work is to establish new necessary and sufficient conditions for the existence of an eventually positive and bounded solution of Eq. (1). Our result, to some extent, improves the corresponding results in [3].

As usual, a solution of Eq. (1) is a continuous function $x(t)$ defined on $[-\mu, \infty)$ such that $y(t) := a(t)x(t) - \sum_{i=1}^{m} b_i(t)x(t-\tau_i)$ is n times differentiable and Eq. (1) holds for all $n \geq 0$. Such a solution $x(t)$ is called an eventually positive solution if there is $T \geq 0$ such that $x(t) > 0$ for $t \geq T$. Here, $\tau = \max_{1\leq i\leq m}\{\tau_i\},\ \sigma = \max_{1\leq j\leq l}\{\sigma_j\},\ \mu = \max\{\tau, \sigma\}$.

Lemma. *Assume* $\sup_{t\geq 0} a(t) < \infty$, *and there exists* $M > 0$ *such that* $\sum_{i=1}^{m} \frac{b_i(t)}{a(t)} \leq M$. *Let* $x(t)$ *be an eventually positive and bounded solution of inequality (2) and*

$$y(t) = a(t)x(t) - \sum_{i=1}^{m} b_i(t)x(t-\tau_i). \tag{9}$$

Then eventually,

$$(-1)^k y^{(k)}(t) \geq 0 \quad (k = 0, 1, \ldots, n), \qquad \lim_{t\to\infty} y^{(k)}(t) = 0 \quad (k = 0, 1, 2, \ldots, n-1). \tag{10}$$

Proof. It is easy to see that $y(t)$ is eventually bounded. From (2), we have $y^{(n)}(t) \geq \sum_{j=1}^{l} p_j(t)f_j\left(t, x(t-\sigma_j)\right) \geq 0$ for sufficiently large t. Now we will prove $y^{(n-1)}(t) \leq 0$ for sufficiently large t. Otherwise $y^{(n-1)}(t) > 0$ for sufficiently large t; since $y^{(n)}(t) \geq 0$, oscillatory behavior of $y^{(n-1)}(t)$ is impossible, so we can assume $\lim_{t\to\infty} y^{(n-1)}(t) = L$ $(0 < L \leq \infty)$. Thus, there exists sufficiently large t_0 such that

$$y^{(n-2)}(t) - y^{(n-2)}(t_0) = \int_{t_0}^{t} y^{(n-1)}(t)\mathrm{d}t \geq (L-\varepsilon)(t-t_0) \to \infty \quad (t \to \infty).$$

That is, $\lim_{t\to\infty} y^{(n-2)}(t) = \infty$. Similarly, $\lim_{t\to\infty} y^{(k)}(t) = \infty$ $(k = 0, 1, \ldots, n-3)$. This contradicts the boundedness of $y(t)$. Therefore, $y^{(n-1)}(t) \leq 0$. Again, since $y^{(n)}(t) \geq 0$ for sufficiently large t, we can assume $\lim_{t\to\infty} y^{(n-1)}(t) = L$ $(L \leq 0)$. Next, we will prove $L = 0$. Otherwise $L < 0$. Furthermore, we have

$$y^{(n-2)}(t) - y^{(n-2)}(t_0) = \int_{t_0}^{t} y^{(n-1)}(t)\mathrm{d}t \leq (L+\varepsilon)(t-t_0) \to -\infty \quad (t \to \infty).$$

So $\lim_{t\to\infty} y^{(n-2)}(t) = -\infty$. Similarly, $\lim_{t\to\infty} y^{(k)}(t) = -\infty$ $(k = 0, 1, \ldots, n-3)$. This contradicts the boundedness of $y(t)$. Therefore, $L = 0$. That is, $\lim_{t\to\infty} y^{(n-1)}(t) = 0$. By repeating the same procedure, we can obtain

$$(-1)^k y^{(k)}(t) \geq 0 \quad (k = 0, 1, \ldots, n), \qquad \lim_{t\to\infty} y^{(k)}(t) = 0 \quad (k = 0, 1, 2, \ldots, n-1).$$

The proof is complete. □

2. Comparison theorem of existence for an eventually positive and bounded solution

In the following, we will give a comparison theorem of existence for an eventually positive and bounded solution.

Theorem. *Assume all the conditions of the lemma hold and that either*

(H_1) $\sum_{i=1}^{m} b_i(t) + \sum_{j=1}^{l} \sigma_j p_j(t) > 0$ *for sufficiently large* t, *or*

(H_2) $\min_{1\leq j\leq l}\{\sigma_j\} > 0$, *and there exists* j_0 $(1 \leq j_0 \leq l)$ *such that* $p_{j_0}(s) \neq 0,\ s \in [t, t+\sigma]$ *for sufficiently large* t.

Then (1) has an eventually positive and bounded solution if and only if (2) has an eventually positive and bounded solution.

Proof. It is clear that an eventually positive and bounded solution of (1) is also an eventually positive and bounded solution of (2). So it suffices to prove that if (2) has an eventually positive and bounded solution $x(t)$, for $t > 0$, then so does Eq. (1). Set

$$y(t) = a(t)x(t) - \sum_{i=1}^{m} b_i(t)x(t-\tau_i).$$

From (10) and integrating (2) from t to ∞, we obtain

$$y^{(n-1)}(t) - y^{(n-1)}(\infty) \geq \int_t^\infty \left[\sum_{j=1}^{l} p_j(s) f_j\left(s, x(s-\sigma_j)\right)\right] \mathrm{d}s.$$

Further

$$y^{(n-1)}(t) \leq -\int_t^\infty \left[\sum_{j=1}^{l} p_j(s) f_j\left(s, x(s-\sigma_j)\right)\right] \mathrm{d}s. \tag{11}$$

By repeating the same procedure and using (10), we have

$$y(t) \geq \int_t^\infty \mathrm{d}t_n \int_{t_n}^\infty \mathrm{d}t_{n-1} \int_{t_{n-1}}^\infty \mathrm{d}t_{n-2} \cdots \int_{t_2}^\infty \left[\sum_{j=1}^{l} p_j(s) f_j\left(s, x(s-\sigma_j)\right)\right] \mathrm{d}s, \quad t \geq 0.$$

Using Tonelli's theorem, we reverse the order of integration and obtain

$$y(t) \geq \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left[\sum_{j=1}^{l} p_j(s) f_j\left(s, x(s-\sigma_j)\right)\right] \mathrm{d}s, \quad t \geq 0.$$

That is,

$$x(t) \geq \frac{1}{a(t)} \sum_{i=1}^{m} b_i(t)x(t-\tau_i) + \frac{1}{a(t)} \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left[\sum_{j=1}^{l} p_j(s) f_j\left(s, x(s-\sigma_j)\right)\right] \mathrm{d}s, \quad t \geq 0. \tag{12}$$

Let $t \geq 0$ be sufficiently large such that (10) holds for $t \geq T$, and $x(t-\mu) > 0$ for $t \geq T$. Set

$$\Omega = \left\{z \in C\left([T-\mu, \infty), R^+\right) : 0 \leq z(t) \leq 1, t \geq T-\mu\right\}$$

and define an operator S on Ω as follows:

$$(Sz)(t) = \begin{cases} \dfrac{t-T+\mu}{\mu}(Sz)(T) + \left(1 - \dfrac{t-T+\mu}{\mu}\right), & T-\mu \leq t < T, \\ \dfrac{1}{x(t)} \left\{ \dfrac{1}{a(t)} \displaystyle\sum_{i=1}^{m} b_i(t)z(t-\tau_i)x(t-\tau_i) \right. & \\ \quad \left. + \dfrac{1}{a(t)} \displaystyle\int_t^\infty \dfrac{(s-t)^{n-1}}{(n-1)!} \left[\sum_{j=1}^{l} p_j(s) f_j\left(s, z(s-\sigma_j)x(s-\sigma_j)\right)\right]\right\} \mathrm{d}s, & t \geq T. \end{cases}$$

By using (12), it is easy to see that S maps Ω into itself. In addition, for any $z \in \Omega$, we have $(Sz)(t) > 0$ $(T-\mu \leq t < T)$. Next, we define the sequence $\{z_k\} \subset \Omega$, where

$$z_0(t) \equiv 1, \quad t \geq T-\mu,$$

and

$$z_{k+1}(t) = (Sz_k)(t), \quad t \geq T-\mu, \ k = 0, 1, \ldots.$$

Then, from (12) and a simple induction, we can easily see that

$$0 \leq z_{k+1}(t) \leq z_k(t) \leq 1, \quad t \geq T-\mu,$$

that is, $Sz \le z$ for all $z \in \Omega$. Set $\lim_{k\to\infty} z_k(t) = z(t),\ t \ge T - \mu$. Then it follows from Lebesgue's dominated convergence theorem that $z(t)$ satisfies

$$z(t) = \begin{cases} \dfrac{t-T+\mu}{\mu}(Sz)(T) + \left(1 - \dfrac{t-T+\mu}{\mu}\right), & T-\mu \le t < T, \\ \dfrac{1}{x(t)}\left\{\dfrac{1}{a(t)}\displaystyle\sum_{i=1}^{m} b_i(t) z(t-\tau_i)x(t-\tau_i)\right. & \\ \quad \left. + \dfrac{1}{a(t)}\displaystyle\int_t^\infty \dfrac{(s-t)^{n-1}}{(n-1)!}\left[\sum_{j=1}^{l} p_j(s) f_j\left(s, z(s-\sigma_j)x(s-\sigma_j)\right)\right]\right\} \mathrm{d}s, & t \ge T. \end{cases}$$

Again, set $\omega(t) = z(t)x(t)$; then $\omega(t)$ is bounded and $\omega(t) > 0 (T - \mu \le t \le T)$. In addition, we have

$$\omega(t) = \frac{1}{a(t)}\sum_{i=1}^{m} b_i(t)\omega(t-\tau_i) + \frac{1}{a(t)}\int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!}\left[\sum_{j=1}^{l} p_j(s) f_j\left(s, \omega(s-\sigma_j)\right)\right] \mathrm{d}s, \quad t \ge T.$$

Thus $\omega(t)$ is a nonnegative and bounded solution of Eq. (1) for $t \ge T$.

Finally, it remains to show that

$$\omega(t) > 0, \quad t \ge T - \mu.$$

Assume that there exists $t^* \ge T - \mu$ such that $\omega(t) > 0\,(T - \mu \le t \le t^*)$ and $\omega(t^*) = 0$. Then we have

$$0 = \omega(t^*) = \frac{1}{a(t^*)}\sum_{i=1}^{m} b_i(t^*)\omega(t^*-\tau_i) + \frac{1}{a(t^*)}\int_{t^*}^\infty \frac{(s-t)^{n-1}}{(n-1)!}\left[\sum_{j=1}^{l} p_j(s) f_j\left(s, \omega(s-\sigma_j)\right)\right] \mathrm{d}s,$$
$$t^* > T,$$

which implies

$$b_i(t^*) = 0, \quad i = 1, 2, \ldots, m,$$

and

$$p_j(s) f_j\left(s, \omega(s-\sigma_j)\right) \equiv 0, \quad j = 1, 2, \ldots, l.$$

This contradicts (H_1) and (H_2). Thus, $\omega(t)$ is an eventually positive and bounded solution of (1). The proof is complete.

In particular, when $m = l = 1$, $f(t, x) \equiv f(x)$, Eq. (1) becomes

$$[a(t)x(t) - b(t)x(t-\tau)]^{(n)} - p(t) f(x(t-\sigma)) = 0, \quad t \ge 0, \tag{13}$$

with the associated differential inequality

$$[a(t)x(t) - b(t)x(t-\tau)]^{(n)} - p(t) f(x(t-\sigma)) \ge 0, \quad t \ge 0. \tag{14}$$

Here $n = 2k, k > 0$ is an integer, $a, b, p \in \left([0,\infty), R^+\right), \tau > 0, \sigma \ge 0$, and $a(t) > 0$, p is not identically zero for sufficiently large t, $f(x)$ is a continuously nondecreasing real function with respect to x defined on R such that $f(x) > 0$, for $x > 0$. Obviously, we can show the following result. □

Corollary. *Suppose* $\sup_{t\ge 0} a(t) < \infty, b(t) < \infty$ *and that either*

(H_1') $b(t) + \sigma p(t) > 0$, *for sufficiently large* t, *or*

(H_2') $\sigma > 0$, *and* $p(s) \ne 0,\ s \in [t, t+\sigma]$, *for sufficiently large* t.

Then (13) has an eventually positive and bounded solution if and only if (14) has an eventually positive and bounded solution.

Remark. Corollary is different from Theorem A.

For example, suppose that σ, q and f satisfy the above assumptions, and $a(t) \equiv 2, \tau = 1, b(t) < \infty$. Then all the conditions of the corollary are satisfied. But it is not easy to find t^* which satisfies the conditions of Theorem A.

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